

# BBGKY hierarchy in scalar QFT.

K. A. Lyakhov<sup>#</sup>

*Sub-department of Quantum Statistics and Field Theory of Physical department, Moscow State University, 119899 Vorobiev Gory, Moscow, Russia*

keywords: *BBGKY hierarchy, Wigner's function, renormalization, scalar field.*

PACS:

This work is dedicated to obtaining of analog of Bogoliubov's chain for the case of complex scalar field in QFT and renormalization problem of obtained equations is discussed.

---

<sup>#</sup> E-mail: [liakhov4@mail.ru](mailto:liakhov4@mail.ru)

## Introduction and sculpting the problem.

In this work attempt to generalise Bogoliubov's chain,[1], on relativistic case, when the particles number is not fixed, proceeding from the QFT dynamic equations is performed. Our consideration will be based on the Wigner's function formalism.

### 1. Notations and preliminaries.

In d-dimensional quantum mechanics, if the system is in a pure state with the wave function  $\Psi$ , the Wigner's function looks as

$$\mathcal{F}(k, x) = \frac{1}{(2\pi\hbar)^d} \int dv e^{\frac{i}{\hbar} + kv} \Psi^*(x + \frac{1}{2}v) \Psi(x - \frac{1}{2}v).$$

In relativistic quantum statistical physics this formula transforms as ,[2]:

$$\mathcal{F}(k, x) = \frac{1}{(2\pi\hbar)^d} \int dv e^{\frac{i}{\hbar} + kv} < \varphi^\dagger(x + \frac{1}{2}v) \varphi(x - \frac{1}{2}v) >, \quad (1.1)$$

where  $\varphi^\dagger$  and  $\varphi$  are the secondary-quantized field operators and averaging is performed by Yuttner-Singh distribution, see below. According to (1.1) the one-particle Wigner's function is the average of filling number, then the norm condition is:

$$\int \mathcal{F}(k, x) dk dx = N \quad (1.2)$$

Let's note through  $z_i = (k_i, x_i)$  the coordinate of i-th particle in relativistic quantum-mechanical phase space.

Let's inductively introduce n-particles Wigner's function in relativistic case:

$$\mathcal{F}_n(z_1 \dots z_n) = N' \int \exp\left(i \sum_{i=1}^n k_i v_i\right) W_{2n}\left(x_1 + \frac{v_1}{2}, \dots, x_n - \frac{v_n}{2}\right) dv_1 \dots dv_n. \quad (1.3)$$

as the Furier-transform of the normal-ordered and averaged operators product:

$$W_{2n}\left(x_1 + \frac{v_1}{2}, \dots, x_n - \frac{v_n}{2}\right) \stackrel{Def}{=} \stackrel{Def}{<} \varphi^\dagger\left(x_1 + \frac{v_1}{2}\right) \dots \varphi^\dagger\left(x_n + \frac{v_n}{2}\right) \varphi\left(x_1 - \frac{v_1}{2}\right) \dots \varphi\left(x_n - \frac{v_n}{2}\right) :>, \quad (1.4)$$

were

$$N' = \frac{1}{(2\pi)^{dn}}.$$

Heretheto we imply  $\hbar = c = k_B = 1$  and field operator  $\hat{\varphi}(x)$  is taken in Haizenberg representation:

$$\varphi(x) = U(t)^\dagger T(\varphi_0(x) U(t)) \quad (1.5)$$

,where  $\varphi_0(x)$  is the solution of the uniform Klein-Gordon's equation and  $U(t)$  is the evolution operator:

$$U = T \exp\left(i \int_{-\infty}^t H_{int}(t) dt\right) \quad (1.6)$$

Averaging-out in (1.2) is produced as to ordinary rules of statistical mechanics:

$$\langle : O : \rangle = \text{Tr}(\rho : O :). \quad (1.7)$$

Here trace is calculated as to any full set of states. The statistical operator  $\rho$  is the relativistic analog of the operator realisation of the grand canonical Gibbs distribution, where the fact, that the particles can produced and disappear, is allowed for by Haizenberg's representation, see(1.3):

$$\rho(t) = \frac{1}{\zeta} \exp(\beta(\mu N(t) - U_\nu P^\nu(t))), \quad (1.8)$$

where  $N(t)$  is operator of number of particles in Haizenberg's representation:

$$N(t) = \int d^3p [a^\dagger(\vec{p}, t)a(\vec{p}, t) + b^\dagger(\vec{p}, t)b(\vec{p}, t)], \quad (1.9)$$

where  $\beta = \frac{1}{T}$ ,  $\mu$  is the chemical potential;  $U_\nu$  is the hydrodynamical velocity [2], and  $\zeta$  is defined by the norm condition:  $\text{Tr}(\rho(t)) = 1$ .

The 4-momentum operator of the system is

$$P^\nu(t) = (H(t), P^k(t))$$

, where  $k = 1, 2, 3$  and

$$H(t) = \int d^3p p^0 [a^\dagger(\vec{p}, t)a(\vec{p}, t) + b^\dagger(\vec{p}, t)b(\vec{p}, t)] \quad (1.10)$$

, where  $p^0 = \sqrt{\vec{p}^2 + m^2}$ . Here  $a^\dagger$  and  $b^\dagger$  are operators of particles and antiparticles production in Haizenberg's representation correspondingly, see(1.5).

Amenably (1.2) the norm condition for s- particle Wigner's function takes form:

$$\int \mathcal{F}_s(k_1, x_1 \dots x_s, k_s) dz_1 \dots dz_s = N^s, \quad (1.11)$$

where  $N$  is the general particles number in the system.

## 2.The model description.

In this work we'll consider the quantum statistical relativistic system, consisting from elementary particles, interacting by Lagrangian of self-interacting complex scalar field:

$$\mathcal{L} = (\partial_\mu \varphi^\dagger \partial_\mu \varphi) m^2 (\varphi^\dagger \varphi) + \lambda (\varphi^\dagger \varphi)^2 \quad (2.1)$$

On the strength of corresponding this model field equations:

$$(\square + m^2)\varphi(x) = \rho(x); \quad (2.2)$$

$$(\square + m^2)\varphi^\dagger(x) = \rho^\dagger(x) \quad (2.3)$$

, where

$$\rho(x) = 2\lambda(\varphi^\dagger(x)\varphi(x))\varphi(x); \quad (2.4)$$

$$\rho^\dagger = 2\lambda(\varphi^\dagger(x)\varphi(x))\varphi^\dagger(x). \quad (2.5)$$

we'll deduce BBGKY hierarchy below.

### 3. The kinetic equation for n-partical Wigner's function.

The Lorentz-invariant kinetic equation for one-particle Wigner's function, as to essence, was already obtained in the monography[2], and look as:

$$k_\mu \partial_\mu \mathcal{F}(z) = i \frac{1}{2(2\pi)^4} \int e^{ikv} < \rho(x + \frac{v}{2}) \varphi(x - \frac{v}{2}) > dv + h.c. \quad (3.1).$$

By substitution like-current field  $\rho$  (2.5) in the given equation we have:

$$\begin{aligned} k_\mu \partial_\mu \mathcal{F}(z) = i\lambda \frac{1}{(2\pi)^4} \int <: \varphi^\dagger(x + \frac{v}{2}) \varphi(x - \frac{v}{2}) \varphi^\dagger(x + \frac{v}{2} + \frac{v'}{2}) \varphi(x + \frac{v}{2} - \frac{v'}{2}) >: \times \\ \times e^{-ik'v'} e^{-ikv} dk' dv dv' + h.c. \end{aligned} \quad (3.2)$$

Here we took into account the integral representation for  $\delta$ -function:

$$\delta(v) = \frac{1}{(2\pi)^4} \int e^{-ikv} dk.$$

So, we have received the first equation of engaging equations:

$$k_\mu \partial_\mu \mathcal{F}(z) = i\lambda \int \mathcal{F}_2 \left( k, x, k', x + \frac{v}{2} \right) e^{-ik'v} dk' dv + h.c. \quad (3.3)$$

The subsequent equations of chain could be found by induction, i.e. it is valid the next

#### Proposition 1

$$\begin{aligned} k_\mu \partial_\mu |_{x_i} \mathcal{F}_n(z_1, \dots, z_n) \\ = i\lambda \int \mathcal{F}_{n+1}(z_1, \dots, z_n, z') e^{-ik'v} dv dk' + h.c., \end{aligned} \quad (3.4)$$

where  $z' = x_i - \frac{1}{2}v, k'$ .

### 4. The renormalization of the chain equations.

As we well know, in QFT in the force of it's locality singularity arise at operator-meaning generalized functions(field operators) production. For their eliminating one resort to renormalization trick in order to radiation corrections have not led to infinite expressions for the couple constant and mass, then in order to search renormalization of the received chain it is essentially to answer on the following question: "How to distinguish renormalized(observed) part of the n-particleal Wigner's function at transition to the semi-QFT (relativistic thermodynamics)?"

For answer on this question advisably to express an arbitrary operator in terms of reduced density matrices describing initial state, then the Hilbert space of the system is defined by initial states, which are identical with asymptotical[2] ones ( $t \rightarrow -\infty$  for infinity remote past).

For the first we shall consider, for simplicity, the case without antiparticles, then the asymptotic pure state is:

$$|p^n\rangle_{\text{in}} = a_{\text{in}}^\dagger(p^n)|0\rangle, \quad n = 0, 1, 2, \dots, \quad (4.1)$$

Here we have used the compact notation:  $p^n$  for set  $p_1^\mu, p_2^\mu, \dots, p_n^\mu$  and  $a_{\text{in}}^\dagger(p^n)$  for asymptotic operators product  $a_{\text{in}}^\dagger(p_1) \dots a_{\text{in}}^\dagger(p_n)$ .

If we cut-off Hamilton operator by volume and tribute the periodic boundary conditions then we have the discrete energetic spectrum corresponding to Hamiltonian:

$$H(t) = \int_{p_{min}}^{p_{max}} d^3p p^0 [a^\dagger(\vec{p}, t) a(\vec{p}, t) + b^\dagger(\vec{p}, t) b(\vec{p}, t)].$$

The number of particles is finite and is eigenvalue of operator

$$N(t) = \int_{p_{min}}^{p_{max}} d^3p [a^\dagger(\vec{p}, t) a(\vec{p}, t) + b^\dagger(\vec{p}, t) b(\vec{p}, t)].$$

According to that the volume of our system is very small quantum effects play essential role. Then energy of system as due to Haizenberg's ambiguity principle is very large and upper integration limit as to momentum  $p_{max}$  could be entered as particle momentum, generated after interaction immediately. The lower integration limit  $p_{min}$  could be entered as particle momentum in the time moment of system evolution accordihg to Haizenberg's ambiguity principle:  $E \sim \frac{\hbar}{\tau}$ , where  $\tau$  is the life-living time of the emmited secondary particles. We stress here, that in this situation statistical considerations are applied, i.e. the number of particles is large, but finite, unlike QFT for which, as we well know, infinitely large degrees of the freedom (field quants) is characteristic. In force that the number of particles is finite the completeness condition is next:

$$\mathbf{1} = \sum_{n=0}^{N(E)} \frac{1}{n!} \int \frac{d^3p^n}{p^{0n}} |p^n\rangle_{in} \langle p^n|, \quad (4.2)$$

where  $N(E)$  is the function of energy of the system. Then the modification of formula for average for arbitrary operator  $O$ , ref.[2], with take into account new completeness condition(4.3)is needed:

$$\langle O \rangle = \sum_{n=0}^{N(E)} \frac{1}{n!} \int d^4x^n d^4k^n O_n(x^n, k^n) \sum_{m=0}^{N(E)} \frac{(-1)^m}{(m!)^2} \int d^4x^m d^4k^m \prod_{j=1}^{n+m} \mathcal{F}_{in}(x_j, k_j). \quad (4.3)$$

Here the integrals as to momenta are cutted off on upper and lower limits according with above said and integrals as to space are cutted off in accordance with the one meson space volume. Multiplier

$\sum_{m=1}^{N(E)} \frac{(-1)^m}{(m!)^2} \int d^4k^m d^4x^m \prod_{j=1}^m \mathcal{F}_{in}(x_j, k_j)$  equals

$$Z = - \langle N_{in}(E) \rangle + \frac{1}{4} (\langle N_{in}(E) \rangle)^2 - \frac{1}{36} (\langle N_{in}(E) \rangle)^3 + \dots + \frac{(-1)^{N(E)}}{(N(E)!)^2} (\langle N_{in}(E) \rangle)^{N(E)}, \quad (4.4)$$

where we used, that

$$\int \mathcal{F}_{in}(k, x) dx dk = \langle N_{in} \rangle \quad (4.5)$$

.  $N_{in}$  is the number of scalar particles operator in initial state. Let's examine the Wigner's function expansion as to the contributions of collisions. After substitution (4.5) into (4.4) we have:

$$\langle O \rangle = \sum_{n=0}^{N(E)} \frac{1}{n!} \int d^4x^n d^4k^n O_n(x^n, k^n) \prod_{j=1}^n \mathcal{F}_{in}(x_j, k_j) \times Z \quad (4.6)$$

Having applicated (4.6) to the one-particle Wigner's function, we have the following expansion:

$$\mathcal{F}(x, k) = \sum_{n=0}^{N(E)} \frac{1}{n!} \int d^4x^n d^4p^n \Psi_n(x^n, p^n | k, x) \prod_{j=1}^n \mathcal{F}_{in}(x + x_j, p_j) Z \quad (4.7)$$

,where

$$\Psi_n(x^n, p^n | k) = N' \int d^4u^n \prod_{j=1}^n e^{iu_j x_j} \langle p^n - \frac{1}{2}u^n | \Psi(k) | p^n + \frac{1}{2}u^n \rangle_{in}. \quad (4.8)$$

,and operator  $\Psi(k)$  is:

$$\Psi(k) = \frac{1}{(2\pi)^4} \int d^4v e^{-ikv} \varphi^\dagger\left(\frac{1}{2}v\right) \varphi\left(-\frac{1}{2}v\right). \quad (4.9)$$

Let us calculate every matrix element  $\Psi_n(x^n, p^n | k)$  separately. Matrix element equals zero at  $n = 0$ , as it have to be for normal-ordered operator.

We have used the completeness condition (3.7) at  $n = 1$ , which leads to the following expression:

$$\begin{aligned} \Psi_1(x, p | k) &= \frac{1}{(2\pi)^4} \sum_{m=0}^{N(E)} \frac{1}{m!} \int \frac{d^3p'^m}{p'^0 m} \delta^{(4)}\left(k + \sum_{j=1}^m p'_j - p\right) \times \\ &\times \int d^4u \exp(iux)_{in} \langle p - \frac{1}{2}u | \varphi^\dagger(0) | p'^m \rangle_{out} \langle p'^m | \varphi(0) | p + \frac{1}{2}u \rangle_{in}. \end{aligned} \quad (4.10)$$

Matrix elements  $\langle p - \frac{1}{2}u | \varphi^\dagger(0) | p'^m \rangle_{out}$  can be calculated by using Jang-Feldman equation.

$$\langle 0 | \varphi(0) | p \rangle_{in} = \frac{1}{\sqrt{2p^0}} \frac{1}{(2\pi)^{3/2}}. \quad (4.11)$$

Finally we have

$$\Psi_1(x, p | k) = \delta^{(4)}(k - p) \delta^{(4)}(x). \quad (4.12)$$

## Proposition 2

Using the results (4.10)-(4.12), the expansion of Wigner's function as to the contributions of collisions is:

$$\begin{aligned} \mathcal{F}(x, k) &= Z \times [\mathcal{F}_{in}(x, k) + \sum_{n=2}^{N(E)} \frac{1}{n!} \int d^4x^n d^4p^n \Psi_n(x^n, p^n | k) \times \\ &\times \prod_{j=1}^n \mathcal{F}_{in}^{(+)}(x + x_j, p_j)]. \end{aligned} \quad (4.13)$$

It is impossible to overlook obvious similarity received expression with virial expansion for the classical distribution function in nonrelativistic statistical physics.

*Remark:* According to above mentioned assumption about initial state of the system, we see, that the first term corresponds to solution of uniform or colisionless kinetic equation:

$$k_\mu \partial_\mu \mathcal{F}_{in}(z) = 0. \quad (4.14)$$

Initial distribution function is the equilibrium relativistic Boze-Einstein distribution function as it discussed above, but without spin:

$$\mathcal{F}_{\text{in}}(k, x) = \frac{1}{(2\pi)^3 [e^{\beta(k_\nu U^\nu(x) - \mu(x))} - 1]} \quad (4.15)$$

The rest of the members allow for the contribution of interparticles interactions. According to (4.5)  $Z$  is the finite value unlike QFT, where it would be diverging quantity and accords to renormalization of wave function:  $\varphi^R(x) = Z_\varphi^{-\frac{1}{2}} \varphi(x)$ , see[6] due to Dyson's transformation. Thus the following statement is valid:

**Proposition 3:**

$$\begin{aligned} k_\mu^n \partial_\mu \Big|_{x_i} F_n(z_1 \dots z_n) = \\ = iN' Z \lambda \int_{k'_{min}}^{k'_{max}} F_{n+1}(z_1 \dots z_n, z') e^{-ik'v} dv dk' + h.c. \end{aligned} \quad (4.16)$$

## 5. The physical illustration.

Received chain may be applicated for the computing of kinetic coefficients in non-linear approach, i.e. when the effect of correlations between particles is essential. By other words, it is necessary to allow for not only interaction of particles with each other, but also the interaction of particles with clusters. For example, such situation could be emerged in description of phase transition in relativistic plasma. The role of clusters play numerous short-living resonances in the elementary particles physics. Conjecturally adequate mathematical tool for non-equilibrium interactions between clusters (resonances) description is the Bogoliubov's chains method(BBGKY). Let's for definity consider the following many body processes:

$$\begin{aligned} pp \rightarrow pp\pi^+\pi^+\pi^-\pi^-\pi^0 \rightarrow p\Delta^{++}\pi^+\pi^-\pi^-\pi^0 \\ pp \rightarrow pn\pi^+\pi^+\pi^+\pi^-\pi^-\pi^- \rightarrow p\Delta^+\pi^+\pi^+\pi^-\pi^-\pi^- \end{aligned} \quad (5.1)$$

and etc. The volume, within that collision occurs and the number of generating particles could be calculated, for example, from multiperipheral model, [5]. In this article it was built the quantitative theory of the multiple mesons production with respect of secondary particles interactions at ultra-high energies. There it was obtained, what the number of particles logarithmically depends on energy  $E$  generated by collision:  $N(E) \sim \ln s$ , where  $E \sim \sqrt{s}$ . Respected that volume of system is Lorentz-contracted  $V \sim \frac{1}{m_\pi^3} \frac{2M}{E}$ , where  $M$  is nucleon mass, maximal  $\pi$ -meson energy-momentum equals

$$p_{max}^\mu = (p_{max}^0 = \sqrt{\vec{p}_{max}^2 + m^2} \sim \frac{E}{N} \sim \frac{\ln s}{\sqrt{s}}, \vec{p}_{max}). \quad (5.2)$$

In the case of nucleon beams collision all emitted energy is exuded in the small volume and spent to pionic stars formation and belated to the binded states formation or resonances, see.(5.1). With the flow of time the system expands and becomes more dilute. I.e. we have a number of, step by step alternating each other descriptions: local or QFT-description, then thermodynamical follows, which replies to nucleon plasma on the early stages of it's formation corresponding to the high energies,

no more than 10GeV or hydrodynamical description [4] for ultra-high energies, then kinetic stage follows, at which correlations accord essential influence on the system dynamics and eventually passes in equilibrium state, corresponding to Yuttner-Singh distribution.

**Summary.**

The kinetic stage of the evolution of the system of scalar particles interacting as  $\lambda(\varphi^\dagger\varphi)^2$ , is described by the chain of equations (4.16).

## References

- [1] N.N.Bogoliubov "The problems of dynamical theory in statistical mechanics", Gostechizdat,1946
- [2] S.de Groot,V.van Leuven "Relativistic kinetic theory", Mir,1983
- [3] D.N.Zubarev "Non-equilibrium statistical thermodynamics",Plenum, New York, 1974
- [4] L.D.Landau, Bull.of Acad.of Sci. of the USSR, Ser.Phys.,**17**,51(1953)
- [5] G.F.Chew,A.Pignotti Phys.Rev **176**2112(1968)
- [6] N.N.Bogoliubov,D.V.Shirkov "Quantum fields", Nauka,1990